Probability Distributions

A Primer – Definitions, Key Distributions & Worked Examples

1 Random Variables and Distributions

Definitions

A **random variable** X is a function assigning a real number to each outcome of a random experiment. It is **discrete** if it takes countably many values, and **continuous** if it takes any value in an interval.

- **PMF** (probability mass function, discrete): $p(x) = P(X = x)$, with $\sum_x p(x) = 1$.
- **PDF** (probability density function, continuous): $f(x) \geq 0$ with $\int_{-\infty}^{\infty} f(x)\,dx = 1$, and $P(a \leq X \leq b) = \int_a^b f(x)\,dx$.
- **CDF** (cumulative distribution function): $F(x) = P(X \leq x)$, for either type.

Expectation and Variance

$$E[X] = \sum_x x\,p(x) \quad \text{(discrete)}, \qquad E[X] = \int_{-\infty}^{\infty} x f(x)\,dx \quad \text{(continuous)}$$

$$\mathrm{Var}(X) = E[(X - E[X])^2] = E[X^2] - (E[X])^2$$

2 Common Discrete Distributions

Bernoulli(p)

Models a single trial with success probability p: $P(X = 1) = p$, $P(X = 0) = 1 - p$.

$$E[X] = p, \qquad \mathrm{Var}(X) = p(1 - p)$$

Binomial(n, p)

Models the number of successes in n independent Bernoulli(p) trials:

$$P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}, \qquad k = 0, 1, \ldots, n$$

$$E[X] = np, \qquad \mathrm{Var}(X) = np(1 - p)$$

Poisson(λ)

Models the number of rare events in a fixed interval, given average rate λ:

$$P(X = k) = \frac{e^{-\lambda}\lambda^k}{k!}, \qquad k = 0, 1, 2, \ldots$$

$$E[X] = \lambda, \qquad \mathrm{Var}(X) = \lambda$$

Worked Example (Binomial)

A fair coin is flipped 10 times. Find $P(X = 6)$ heads.

$$P(X = 6) = \binom{10}{6}(0.5)^6(0.5)^4 = 210 \times (0.5)^{10} = \frac{210}{1024} \approx 0.205$$

3 Common Continuous Distributions

Uniform(a, b)

Equal density across the interval:

$$f(x) = \frac{1}{b-a}, \quad a \le x \le b, \qquad E[X] = \frac{a+b}{2}, \qquad \mathrm{Var}(X) = \frac{(b-a)^2}{12}$$

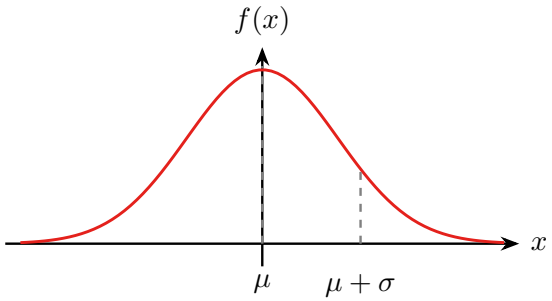
Exponential(λ)

Models waiting time until the next event of a Poisson process with rate λ:

$$f(x) = \lambda e^{-\lambda x}, \quad x \ge 0, \qquad E[X] = \frac{1}{\lambda}, \qquad \mathrm{Var}(X) = \frac{1}{\lambda^2}$$

Normal (Gaussian) $N(\mu, \sigma^2)$

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \qquad E[X] = \mu, \qquad \mathrm{Var}(X) = \sigma^2$$



The Normal (bell) curve

Worked Example (Normal)

Heights are $N(170, 100)$ (mean 170cm, variance 100, so $\sigma = 10$cm). Find $P(160 < X < 180)$.
This range is exactly $\mu \pm \sigma$, so by the Empirical (68–95–99.7) Rule:

$$P(160 < X < 180) \approx 68\%.$$

Worked Example (Exponential)

The waiting time for a bus is Exponential with mean 10 minutes ($\lambda = 1/10$). Find $P(X > 15)$.

$$P(X > 15) = e^{-\lambda \cdot 15} = e^{-1.5} \approx 0.223$$