

Metric Spaces

A Primer – Definitions, Key Theorems & Worked Examples

1 Metrics and Metric Spaces

Definition

A **metric** on a set X is a function $d : X \times X \rightarrow [0, \infty)$ satisfying, for all $x, y, z \in X$:

1. $d(x, y) = 0 \iff x = y$;
2. **Symmetry**: $d(x, y) = d(y, x)$;
3. **Triangle inequality**: $d(x, z) \leq d(x, y) + d(y, z)$.

The pair (X, d) is a **metric space**.

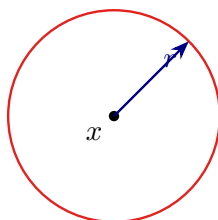
Examples of Metrics

- **Euclidean metric** on $\mathbb{R}^n$: $d(x, y) = \sqrt{\sum_{i=1}^n (x_i - y_i)^2}$.
- **Taxicab (Manhattan) metric**: $d(x, y) = \sum_{i=1}^n |x_i - y_i|$.
- **Discrete metric**: $d(x, y) = 0$ if $x = y$, else 1.
- **Sup metric** on continuous functions $C[a, b]$: $d(f, g) = \sup_{t \in [a, b]} |f(t) - g(t)|$.

2 Open Balls and Open Sets

Definition

The **open ball** of radius $r > 0$ centered at x is $B(x, r) = \{y \in X : d(x, y) < r\}$. A set $U \subseteq X$ is **open** if every point of U has some open ball around it contained entirely in U .



$$B(x, r) = \{y : d(x, y) < r\}$$

3 Convergence and Completeness

Convergent Sequences

A sequence (x_n) in (X, d) **converges** to x if $d(x_n, x) \rightarrow 0$ as $n \rightarrow \infty$: for every $\varepsilon > 0$, there exists N such that $d(x_n, x) < \varepsilon$ for all $n \geq N$.

Cauchy Sequences

A sequence (x_n) is **Cauchy** if for every $\varepsilon > 0$ there is N such that $d(x_n, x_m) < \varepsilon$ for all $n, m \geq N$ – the terms get arbitrarily close to *each other*, not necessarily to a fixed limit.

Complete Metric Space

(X, d) is **complete** if every Cauchy sequence in X converges to a point *in* X .

A Space That Is Not Complete

$\mathbb{Q}$ (the rationals) with the usual metric $d(x, y) = |x - y|$ is *not* complete: the sequence of decimal truncations of $\sqrt{2}$ (i.e. 1, 1.4, 1.41, 1.414, ...) is Cauchy in $\mathbb{Q}$, but its limit $\sqrt{2}$ is not rational. By contrast, $\mathbb{R}$ *is* complete – this is one of the most important facts distinguishing $\mathbb{R}$ from $\mathbb{Q}$.

4 Continuity in Metric Spaces

ε - δ Definition

$f : (X, d_X) \rightarrow (Y, d_Y)$ is **continuous at** x_0 if for every $\varepsilon > 0$ there is $\delta > 0$ such that

$$d_X(x, x_0) < \delta \implies d_Y(f(x), f(x_0)) < \varepsilon.$$

5 Compactness in Metric Spaces

Sequential Compactness

In a metric space, X is compact (every open cover has a finite subcover) if and only if X is **sequentially compact**: every sequence in X has a subsequence converging to a point of X .

Note: This equivalence is special to metric spaces – in general topological spaces, compactness and sequential compactness can differ. It is why, in analysis, “compact” is so often checked via sequences (e.g. Bolzano–Weierstrass) rather than open covers.

6 The Contraction Mapping Theorem

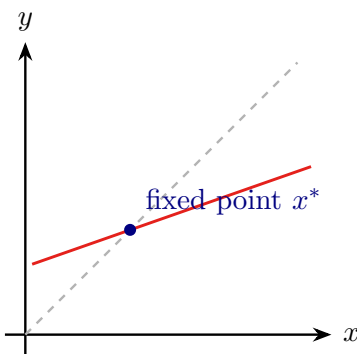
Contraction

A map $f : X \rightarrow X$ on a metric space is a **contraction** if there is a constant $0 \leq k < 1$ such that

$$d(f(x), f(y)) \leq k d(x, y) \quad \text{for all } x, y \in X.$$

Banach Fixed Point Theorem

If (X, d) is a **complete** metric space and $f : X \rightarrow X$ is a contraction, then f has exactly one **fixed point** x^* (i.e. $f(x^*) = x^*$), and for any starting point x_0 , the sequence $x_{n+1} = f(x_n)$ converges to x^* .



Contraction f meets $y = x$ at a unique fixed point

Worked Example

Let $f(x) = \frac{1}{2}x + 3$ on $\mathbb{R}$ (complete). Since $|f(x) - f(y)| = \frac{1}{2}|x - y|$, f is a contraction with $k = \frac{1}{2}$. Solve for the fixed point directly:

$$x^* = \frac{1}{2}x^* + 3 \Rightarrow \frac{1}{2}x^* = 3 \Rightarrow x^* = 6.$$

Starting from any x_0 , iterating $x_{n+1} = \frac{1}{2}x_n + 3$ converges to 6.