

Measure Theory

A Primer – Definitions, Key Theorems & Worked Examples

1 σ -Algebras and Measurable Spaces

Definition

Let X be a set. A collection $\mathcal{F}$ of subsets of X is a **σ -algebra** if:

1. $X \in \mathcal{F}$;
2. $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$ (closed under complements);
3. $A_1, A_2, \dots \in \mathcal{F} \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$ (closed under countable unions).

The pair $(X, \mathcal{F})$ is called a **measurable space**, and sets in $\mathcal{F}$ are **measurable sets**.

Note: Since $\mathcal{F}$ is closed under complements and countable unions, De Morgan's laws show it is also closed under countable intersections: $\bigcap_n A_n = (\bigcup_n A_n^c)^c$.

Borel σ -algebra

The **Borel σ -algebra** $\mathcal{B}(\mathbb{R})$ is the smallest σ -algebra on $\mathbb{R}$ containing all open intervals. It contains every open set, every closed set, and every set built from these by countably many unions, intersections, and complements.

2 Measures

Definition

A **measure** on $(X, \mathcal{F})$ is a function $\mu : \mathcal{F} \rightarrow [0, \infty]$ such that:

1. $\mu(\emptyset) = 0$;
2. (**Countable additivity**) if $A_1, A_2, \dots \in \mathcal{F}$ are pairwise disjoint, then

$$\mu \left(\bigcup_{n=1}^{\infty} A_n \right) = \sum_{n=1}^{\infty} \mu(A_n).$$

The triple $(X, \mathcal{F}, \mu)$ is a **measure space**. If $\mu(X) = 1$, μ is a **probability measure**.

Examples of Measures

- **Counting measure:** $\mu(A) = |A|$ (number of elements), possibly ∞ .
- **Lebesgue measure** m on $\mathbb{R}$: assigns to each interval (a, b) its length $b - a$, and extends (via outer measure and Carathéodory's extension theorem) to all Borel sets and more.
- **Dirac measure** δ_{x_0} : $\delta_{x_0}(A) = 1$ if $x_0 \in A$, else 0.

Basic Properties of Measures

For a measure μ on $(X, \mathcal{F})$:

- **Monotonicity:** $A \subseteq B \Rightarrow \mu(A) \leq \mu(B)$.
- **Countable subadditivity:** $\mu(\bigcup_n A_n) \leq \sum_n \mu(A_n)$.
- **Continuity from below:** if $A_1 \subseteq A_2 \subseteq \dots$, then $\mu(\bigcup_n A_n) = \lim_{n \rightarrow \infty} \mu(A_n)$.
- **Continuity from above:** if $A_1 \supseteq A_2 \supseteq \dots$ and $\mu(A_1) < \infty$, then $\mu(\bigcap_n A_n) = \lim_{n \rightarrow \infty} \mu(A_n)$.

3 Measurable Functions

Definition

A function $f : X \rightarrow \mathbb{R}$ is **measurable** (with respect to $\mathcal{F}$) if for every $a \in \mathbb{R}$,

$$\{x \in X : f(x) > a\} \in \mathcal{F}.$$

Note: Sums, products, and pointwise limits of measurable functions are measurable – this is what makes the class of measurable functions robust enough to build an integration theory on.

4 The Lebesgue Integral

Construction, in Three Stages

1. For a **simple function** $s = \sum_{i=1}^n c_i \mathbf{1}_{A_i}$ (finitely many values c_i on measurable sets A_i), define

$$\int s \, d\mu = \sum_{i=1}^n c_i \mu(A_i).$$

2. For $f \geq 0$ measurable, define $\int f \, d\mu = \sup \left\{ \int s \, d\mu : 0 \leq s \leq f, \, s \text{ simple} \right\}$.

3. For general f , write $f = f^+ - f^-$ (positive and negative parts) and define $\int f \, d\mu = \int f^+ \, d\mu - \int f^- \, d\mu$, provided not both integrals are infinite.

Example

Let $\mu = m$ (Lebesgue measure) and $f = \mathbf{1}_{\mathbb{Q} \cap [0,1]}$ (indicator of the rationals in $[0,1]$). Since $\mathbb{Q}$ is countable, $m(\mathbb{Q} \cap [0,1]) = 0$, so

$$\int_0^1 f \, dm = 0.$$

Note: This function is famously not Riemann integrable (it oscillates between 0 and 1 on every subinterval), yet it is trivially Lebesgue integrable – a first hint of why measure theory extends the reach of integration.

5 Convergence Theorems

Monotone Convergence Theorem (MCT)

If $0 \leq f_1 \leq f_2 \leq \dots$ is an increasing sequence of measurable functions converging pointwise to f , then

$$\lim_{n \rightarrow \infty} \int f_n \, d\mu = \int f \, d\mu = \int \lim_{n \rightarrow \infty} f_n \, d\mu.$$

Fatou's Lemma

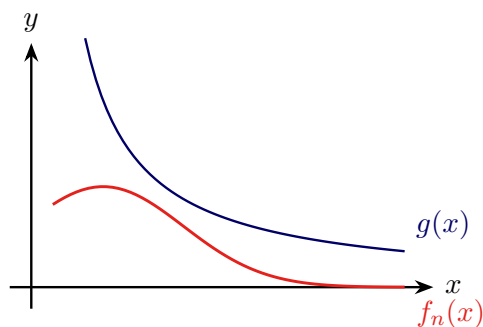
For any sequence of non-negative measurable functions f_n ,

$$\int \liminf_{n \rightarrow \infty} f_n \, d\mu \leq \liminf_{n \rightarrow \infty} \int f_n \, d\mu.$$

Dominated Convergence Theorem (DCT)

If $f_n \rightarrow f$ pointwise and $|f_n| \leq g$ for all n , where g is integrable ($\int g \, d\mu < \infty$), then

$$\lim_{n \rightarrow \infty} \int f_n \, d\mu = \int f \, d\mu.$$



$|f_n| \leq g$: the dominating function g traps every f_n

Why DCT Matters

DCT lets you swap a limit and an integral – something that fails in general. For example, $f_n(x) = n \mathbf{1}_{(0,1/n)}$ on $[0, 1]$ satisfies $f_n \rightarrow 0$ pointwise, yet $\int f_n = 1$ for every n : there is no single integrable dominating function g , so DCT does not apply, and indeed $\lim \int f_n \neq \int \lim f_n$.