

Topology

A Primer – Definitions, Key Theorems & Worked Examples

1 Topological Spaces

Definition

A **topology** on a set X is a collection τ of subsets of X (called **open sets**) such that:

1. $\emptyset, X \in \tau$;
2. Any union (finite or infinite) of sets in τ is in τ ;
3. Any *finite* intersection of sets in τ is in τ .

The pair (X, τ) is a **topological space**. A set is **closed** if its complement is open.

Examples of Topologies

- **Standard topology on $\mathbb{R}$:** open sets are (arbitrary unions of) open intervals (a, b) .
- **Discrete topology:** $\tau =$ all subsets of X (every set is open).
- **Indiscrete (trivial) topology:** $\tau = \{\emptyset, X\}$ only.
- **Metric topology:** for a metric space (X, d) , declare U open if every point of U has an open ball around it contained in U .

Basis for a Topology

A collection $\mathcal{B}$ of subsets of X is a **basis** if every open set can be written as a union of elements of $\mathcal{B}$. Equivalently, τ is generated by $\mathcal{B}$ if $U \in \tau$ iff for every $x \in U$ there is $B \in \mathcal{B}$ with $x \in B \subseteq U$.

2 Continuity

Definition

A function $f : (X, \tau_X) \rightarrow (Y, \tau_Y)$ is **continuous** if $f^{-1}(V) \in \tau_X$ for every open set $V \in \tau_Y$.

Note: This purely set-theoretic definition of continuity, using only open sets and no distances, is exactly equivalent to the familiar ε - δ definition when X and Y are metric spaces with their metric topologies.

Homeomorphism

A **homeomorphism** is a continuous bijection $f : X \rightarrow Y$ whose inverse f^{-1} is also continuous. If such an f exists, X and Y are **homeomorphic** (topologically identical, “same shape”).

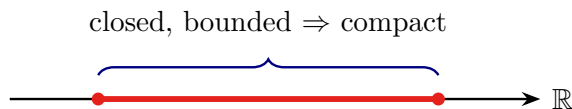
3 Compactness

Definition

An **open cover** of X is a collection of open sets whose union is X . A space X is **compact** if every open cover has a **finite subcover** – finitely many of the sets already cover X .

Heine–Borel Theorem

A subset of $\mathbb{R}^n$ (with the standard topology) is compact if and only if it is **closed and bounded**.



Continuous Image of a Compact Set is Compact

If $f : X \rightarrow Y$ is continuous and X is compact, then $f(X)$ is compact.

Note: This single fact is what powers the Extreme Value Theorem from calculus: a continuous real-valued function on a closed, bounded interval attains a maximum and minimum, because the continuous image of that compact interval is itself compact (closed and bounded), hence contains its own supremum and infimum.

4 Connectedness

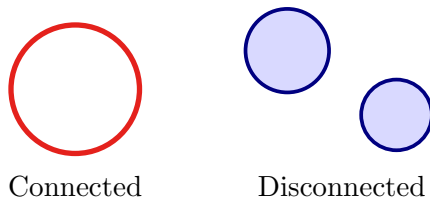
Definition

A space X is **disconnected** if it can be written as $X = U \cup V$ for disjoint, non-empty open sets U, V . Otherwise, X is **connected**.

Continuous Image of a Connected Set is Connected

If $f : X \rightarrow Y$ is continuous and X is connected, then $f(X)$ is connected.

Note: This is the topological reason behind the Intermediate Value Theorem: the continuous image of a connected interval cannot “jump over” any value between $f(a)$ and $f(b)$, since a gap would disconnect the image.



5 Separation Axioms (Brief)

Hausdorff Spaces (T_2)

A space X is **Hausdorff** if any two distinct points $x \neq y$ have disjoint open neighborhoods $U \ni x$, $V \ni y$ with $U \cap V = \emptyset$.

Note: Every metric space is automatically Hausdorff (take balls of radius $d(x, y)/2$ around each point). Hausdorff-ness guarantees limits of sequences are unique – a property one usually takes for granted.

Worked Example

Show the indiscrete topology on $X = \{a, b\}$ (where $\tau = \{\emptyset, X\}$) is *not* Hausdorff.

Solution. The only open sets are $\emptyset$ and X itself. Any open set containing a must be X , which also contains b . So no open neighborhood of a can avoid b – the required disjoint neighborhoods do not exist. Hence (X, τ) is not Hausdorff.