

Stochastic Processes

A Primer – Definitions, Key Theorems & Worked Examples

1 What Is a Stochastic Process?

Definition

A **stochastic process** is a collection of random variables $\{X_t\}_{t \in T}$, indexed by a set T (the **index set**, often time), all taking values in some **state space** S .

- If T is countable (e.g. $T = \{0, 1, 2, \dots\}$), the process is in **discrete time**.
- If T is an interval (e.g. $T = [0, \infty)$), the process is in **continuous time**.

2 Markov Chains

The Markov Property

A discrete-time process $\{X_n\}$ has the **Markov property** if the future depends on the past only through the present:

$$P(X_{n+1} = j \mid X_n = i, X_{n-1} = i_{n-1}, \dots, X_0 = i_0) = P(X_{n+1} = j \mid X_n = i).$$

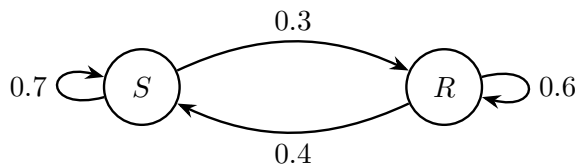
A process with this property and a countable state space is a **Markov chain**.

Transition Matrix

For a Markov chain with states $1, \dots, k$, the **transition matrix** P has entries

$$P_{ij} = P(X_{n+1} = j \mid X_n = i),$$

where every row sums to 1 (from state i , the chain must go *somewhere*).



Note: This two-state “weather” chain has states S (sunny) and R (rainy). From S : stay sunny with probability 0.7, switch to rain with probability 0.3. From R : switch to sunny with probability 0.4, stay rainy with probability 0.6.

Chapman–Kolmogorov Equation

If P is the one-step transition matrix, the n -step transition probabilities are given by the matrix power P^n :

$$P(X_n = j \mid X_0 = i) = (P^n)_{ij}.$$

Stationary Distribution

A probability distribution π (a row vector with $\pi_i \geq 0$, $\sum_i \pi_i = 1$) is **stationary** for P if

$$\pi P = \pi.$$

Once a chain reaches its stationary distribution, it stays there: π describes the long-run fraction of time spent in each state.

Worked Example: Stationary Distribution of the Weather Chain

Using the transition matrix from the diagram above,

$$P = \begin{pmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{pmatrix},$$

solve $\pi P = \pi$ for $\pi = (\pi_S, \pi_R)$ with $\pi_S + \pi_R = 1$:

$$0.7\pi_S + 0.4\pi_R = \pi_S \Rightarrow 0.4\pi_R = 0.3\pi_S \Rightarrow \pi_R = \frac{3}{4}\pi_S.$$

Substituting into $\pi_S + \pi_R = 1$: $\pi_S + \frac{3}{4}\pi_S = 1 \Rightarrow \frac{7}{4}\pi_S = 1 \Rightarrow \pi_S = \frac{4}{7}$, $\pi_R = \frac{3}{7}$.
So in the long run, it is sunny $\frac{4}{7} \approx 57\%$ of the time and rainy $\frac{3}{7} \approx 43\%$ of the time.

3 The Poisson Process

Definition

A **Poisson process** $\{N(t)\}_{t \geq 0}$ with rate $\lambda > 0$ counts events occurring randomly in continuous time such that:

- $N(0) = 0$;
- the numbers of events in disjoint time intervals are independent;
- the number of events in any interval of length t is $\text{Poisson}(\lambda t)$: $P(N(t) = k) = \frac{e^{-\lambda t}(\lambda t)^k}{k!}$.

Note: The waiting time between consecutive events of a Poisson process is $\text{Exponential}(\lambda)$ – this is exactly the connection mentioned in the Probability Distributions primer.

Worked Example

Customers arrive at a shop as a Poisson process at rate $\lambda = 4$ per hour. Find the probability of exactly 2 arrivals in a 30-minute period.

Here $t = 0.5$ hours, so $\lambda t = 2$:

$$P(N(0.5) = 2) = \frac{e^{-2} 2^2}{2!} = \frac{4e^{-2}}{2} = 2e^{-2} \approx 0.271.$$

4 Brownian Motion (Brief)

Definition

Standard Brownian motion $\{B(t)\}_{t \geq 0}$ is a continuous-time process with:

- $B(0) = 0$;
- independent increments: $B(t) - B(s)$ is independent of the process up to time s , for $t > s$;
- $B(t) - B(s) \sim N(0, t - s)$ (normally distributed increments);
- continuous sample paths (as a function of t , with probability 1).

Note: Brownian motion is the continuous-time, continuous-state analogue of a simple random walk, and is the foundational building block of continuous-time finance models (e.g. the Black-Scholes option pricing model).