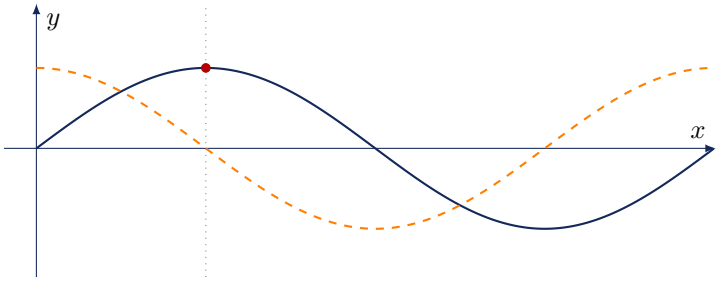
Real Analysis: A Problem Companion

Based on the Problem Sheet of A. B. Raha Sir

July 19, 2026

How to use this document. The original 25 problems have been grouped by topic, each with a short theory refresher, occasional hints, and a companion "bonus" fact that wasn't on the original sheet but pairs naturally with it. There are also a few (harmless) math puns along the way — consider them the ε of comic relief in an otherwise serious δ of real analysis.



The series $S(x) = \sin x$ (solid) and $C(x) = \cos x$ (dashed) — see Section 11.

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1 Sets, Order, and the Architecture of $\mathbb{R}$

Refresher

Two sets A, B are *equivalent* (or equinumerous) if there is a bijection $A \to B$. A set is *finite* if it is equivalent to $\{1, 2, \ldots, n\}$ for some $n \in \mathbb{N} \cup \{0\}$; otherwise it is *infinite*. The order axioms on $\mathbb{R}$ say: for $a > 0$, a has a multiplicative inverse $1/a > 0$, and multiplying an inequality by a positive number preserves its direction, while multiplying by a negative number reverses it.

Problem 1.1. *No finite, non-empty set can be equivalent to any proper subset of it.*

Problem 1.2. *Let $a, b \in \mathbb{R}$. Show that*

$$0 < \frac{1}{b} < \frac{1}{a} \quad \text{if} \quad 0 < a < b, \qquad \text{and} \qquad \frac{1}{b} < \frac{1}{a} < 0 \quad \text{if} \quad a < b < 0.$$

Hint

For Problem 2, multiply the inequality $a < b$ by the positive quantity $\dfrac{1}{ab}$ — but first check the sign of ab in each case before you multiply!

Problem 1.3. *Let $M \subset \mathbb{Z}$ be a non-empty set of integers bounded below. Show that M has a smallest element.*

Companion result — not on the original sheet

Problem 3 is the "discrete cousin" of the *completeness axiom*. The continuous analogue is: *every non-empty subset of $\mathbb{R}$ bounded below has an infimum* (greatest lower bound). The integers are special because that infimum is always attained — gaps of size 1 between consecutive integers prevent the infimum from being a limit point that is never reached. Try proving the well-ordering principle for $\mathbb{N}$ first (every non-empty subset of $\mathbb{N}$ has a least element), then shift by an integer to handle $M \subset \mathbb{Z}$.

:) Pun break

Why did the finite set refuse to date a proper subset? Because it already knew: "we're just not equivalent."

2 Compactness in $\mathbb{R}$

Refresher

A set $K \subset \mathbb{R}$ is *compact* if every open cover of K has a finite subcover. The **Heine–Borel Theorem** tells us that, in $\mathbb{R}$, K is compact $\iff$ K is closed and bounded.

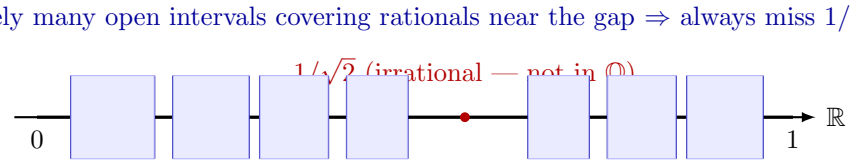
Problem 2.1. *Using the definition of compactness (open covers), show that the set $\mathbb{Q}$ of rationals is not compact.*

Hint

Don't reach for Heine–Borel directly if the point of the exercise is to work with covers. Try covering $\mathbb{Q}$ by open intervals of shrinking length around each rational, engineered so that no finite subcollection can cover all of $\mathbb{Q}$ — e.g., use the fact that $\mathbb{Q}$ is unbounded, or that it's not closed (it's missing irrational limit points such as $\sqrt{2}$).

Companion fact

$\mathbb{Q}$ fails compactness for *two independent reasons*: it is unbounded (consider $\bigcup_n (-n, n) \cap \mathbb{Q}$), and even the bounded piece $\mathbb{Q} \cap [0,1]$ is not compact, since it is not closed in $\mathbb{R}$ (irrational points like $1/\sqrt{2}$ are limit points not in the set). This double failure is a nice illustration of why *both* hypotheses in Heine–Borel are needed.



Any finite collection of open intervals covering $\mathbb{Q} \cap [0,1]$ must still leave irrational gaps like $1/\sqrt{2}$ exposed — a visual hint for why $\mathbb{Q}$ can never be compact.

3 Sequences, Cluster Points, lim sup and lim inf

Refresher

A point c is a *cluster point* (or subsequential limit) of $\{x_n\}$ if some subsequence of $\{x_n\}$ converges to c. Writing

$$S_n = \sup\{x_m : m \geq n\}, \qquad \limsup_{n\to\infty} x_n = \lim_{n\to\infty} S_n = \inf_n S_n,$$

and similarly with inf for lim inf. Both always exist in the extended reals $[-\infty, \infty]$.

Problem 3.1. *Let $\{x_n, n \in \mathbb{N}\}$ be a sequence of real numbers. Let $S_n = \{x \in \mathbb{R} : x = x_m$ for some $m \geq n\}$. Let C denote the set of cluster points of $\{x_n\}$. Show that*

$$C = \bigcap_{n=1}^{\infty} \overline{S_n},$$

and hence deduce that C is closed in $\mathbb{R}$. If K and L stand respectively for $\limsup_{n\to\infty} x_n$ and $\liminf_{n\to\infty} x_n$, then prove that $K, L \in C$ and, if $a \in C$, then $L \leq a \leq K$.

Problem 3.2. *Does* $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}$ *converge?*

Hint

This is a geometric series with ratio $r = 1/2 < 1$; it converges to 1. (A quick sanity check via partial sums: $S_N = 1 - 1/2^N \to 1$.)

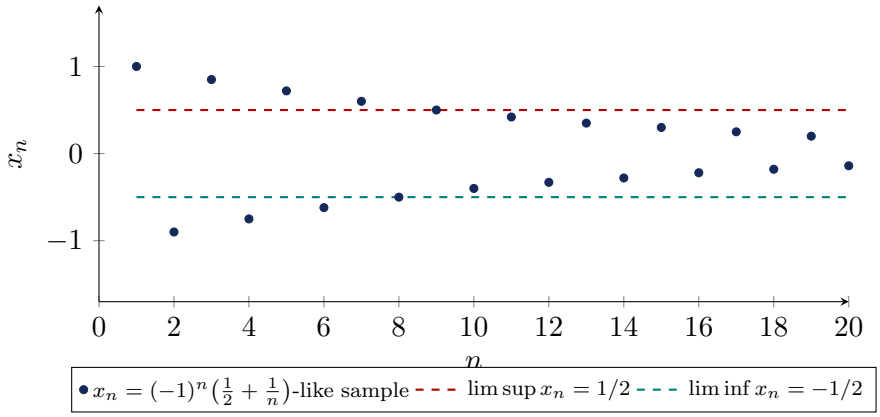
Problem 3.3. *For any sequence* $\{x_n\}$*, show that* $\liminf_{n\to\infty} x_n = -\limsup_{n\to\infty}(-x_n)$.

:) Pun break

lim sup and lim inf walked into a bar. The bartender said, "we don't serve unbounded sequences here." lim sup replied, "that's fine, I'll just take the inf of the sups."

Companion result

Bolzano–Weierstrass Theorem (not on the original sheet, but the natural partner to Problem 5): every bounded sequence in $\mathbb{R}$ has a convergent subsequence, i.e., $C \neq \emptyset$ whenever $\{x_n\}$ is bounded. Combined with Problem 5, this tells you C is a non-empty, closed, bounded set — hence compact! — sandwiched between $L = \min C$ and $K = \max C$.



An oscillating sequence squeezing down toward two distinct cluster points $\pm 1/2$*: the envelope lines are exactly* lim sup *and* lim inf.

4 Absolute Values and Inequalities

Problem 4.1. *If* $a, b \in \mathbb{R}$*, prove that* $|ab| = |a|\,|b|$.

Hint

Case-split on the signs of a and b (four cases, but two pairs are symmetric), or use $|x| = \sqrt{x^2}$ together with $(ab)^2 = a^2b^2$ and the fact that $\sqrt{\cdot}$ is multiplicative on non-negatives.

:) Pun break

The triangle inequality and $|ab| = |a||b|$ went to couples' therapy. The therapist said, "you two need to learn the difference between $\leq$ and $=$."

5 Infinite Series: Convergence, Conditional Convergence, and Sums

Refresher

A series $\sum a_n$ converges *absolutely* if $\sum |a_n|$ converges; it converges *conditionally* if $\sum a_n$ converges but $\sum |a_n|$ does not. Writing $a_n^+ = \max(a_n, 0)$ and $a_n^- = \max(-a_n, 0)$ splits a series into its positive and negative parts.

Problem 5.1. *Prove: if a series is conditionally convergent, the series obtained from its positive terms is not convergent, and the series obtained from its negative terms is not convergent.*

Companion result — a favorite "wow" fact

Riemann Rearrangement Theorem (a natural sequel to Problem 9): if $\sum a_n$ converges conditionally, then for *any* target value $L \in [-\infty, \infty]$, there is a rearrangement of the terms of $\sum a_n$ that converges to L. This is exactly what Problem 9 sets up: since both the positive-term and negative-term subseries diverge (to $+\infty$ and $-\infty$ respectively), you have unlimited "ammunition" of both signs to aim the partial sums wherever you like.

Problem 5.2. *Does the series* $1 + \dfrac{1}{3} + \dfrac{1}{5} + \cdots = \displaystyle\sum_{n=1}^{\infty} \frac{1}{2n-1}$ *converge?*

Hint

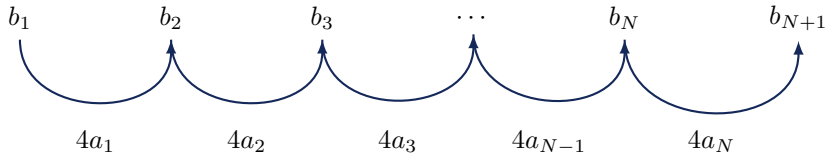
Compare with the harmonic series $\sum 1/n$ via the limit comparison test: $\dfrac{1/(2n-1)}{1/n} \to \dfrac{1}{2}$. Since $\sum 1/n$ diverges, so does this series.

Problem 5.3. *Find the sum of the series* $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)(n+3)(n+4)}$.

(Hint: look at $\dfrac{1}{n(n+1)(n+2)(n+3)} - \dfrac{1}{(n+1)(n+2)(n+3)(n+4)}$ and find a relation with the terms of the series.)

Hint

This is a telescoping series. If $b_n = \dfrac{1}{n(n+1)(n+2)(n+3)}$, check that $b_n - b_{n+1} = \dfrac{4}{n(n+1)(n+2)(n+3)(n+4)}$, so the original series equals $\dfrac{1}{4}(b_1 - \lim_{n\to\infty} b_n) = \dfrac{1}{4} \cdot \dfrac{1}{24} = \dfrac{1}{96}$.



Each arc represents a term $4a_n = b_n - b_{n+1}$ *of the original series. Summing all the arcs telescopes down to just* $b_1 - \lim_N b_{N+1}$.

:) Pun break

Telescoping series don't actually collapse — they just know how to *cancel plans* with their neighbors.

6 Cauchy Sequences

Refresher

$\{x_n\}$ is *Cauchy* if for every $\varepsilon > 0$ there is N such that $|x_n - x_m| < \varepsilon$ for all $n, m \geq N$. In $\mathbb{R}$, Cauchy $\iff$ convergent (completeness). Uniform continuity is the key tool that lets continuous functions "transport" the Cauchy property.

Problem 6.1. *Let $\{x_n, n \in \mathbb{N}\}$ be a Cauchy sequence in a subset $S \subset \mathbb{R}$. Let $x_0 \in \mathbb{R}$ be a cluster point of the sequence. Show that $x_n \to x_0$ as $n \to \infty$.*

Problem 6.2. *Let A be a non-empty subset of $\mathbb{R}$. Suppose $f : A \to \mathbb{R}$ is uniformly continuous on A. If $\{x_n, n \in \mathbb{N}\}$ is a Cauchy sequence in A, show that $\{f(x_n), n \geq 1\}$ is a Cauchy sequence in $\mathbb{R}$.*

Problem 6.3. *Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous function. Suppose $\{x_n, n \geq 1\}$ is a Cauchy sequence in $\mathbb{R}$. Show that $\{f(x_n), n \in \mathbb{N}\}$ is also a Cauchy sequence in $\mathbb{R}$.*

Hint

Compare Problems 13 and 14 carefully: in 13, *mere continuity* is not enough in general — uniform continuity is what's used. In 14, f is continuous on all of $\mathbb{R}$ and the sequence is Cauchy, hence convergent (completeness of $\mathbb{R}$); continuity at the limit point does the rest. This contrast is exactly why Problem 22 (below) matters.

:) Pun break

A Cauchy sequence walks into a bar and orders a drink. The bartender asks, "how close do you want it?" The sequence says, "eventually, as close as you like."

7 Limits and Continuity of Functions

Problem 7.1. *Let f and g be two functions defined on a set $A \subset \mathbb{R}$ into $\mathbb{R}$, and let a be an accumulation point of A. Suppose $\lim_{x \to a} f(x) = \beta$, $\lim_{x \to a} g(x) = \delta$ exist. Show that $\lim_{x \to a} f(x)g(x) = \beta\delta$.*

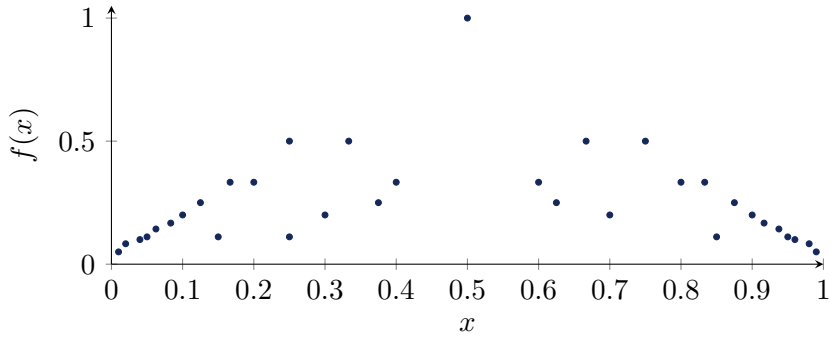
Problem 7.2. *Let $I = \{x \in \mathbb{R} : 0 < x \leq 1\}$ and $f : I \to \mathbb{R}$ be defined by*

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ \dfrac{1}{n} & \text{if } x = \dfrac{m}{n} \text{ in lowest terms, } m, n \in \mathbb{N} \end{cases}$$

Prove that f is continuous at every irrational in I and f is not continuous at each rational in I.

Meet the celebrity

This function is famous enough to have a nickname: **Thomae's function**, also called the *popcorn function* or *raindrop function* (the graph looks like scattered droplets that thin out as you approach the x-axis). It's a beautiful counterexample showing that a function can be continuous at *every* irrational and discontinuous at *every* rational — the reverse (continuous only on $\mathbb{Q}$) is actually **impossible** for any function on $\mathbb{R}$, by the Baire Category Theorem!



Thomae's function on $(0,1]$: height $1/n$ at each rational m/n in lowest terms, and 0 (invisible, dense) at every irrational — the "popcorn" that thins out as n grows.

Problem 7.3. *Let $I \subset \mathbb{R}$ be an open interval. Construct an example of a continuous function $f : I \to \mathbb{R}$ such that the image $f(I)$ of f is not an open interval of $\mathbb{R}$.*

Hint

Try a non-injective function on an open interval, e.g. $f(x) = x^2$ on $I = (-1, 1)$, which gives image $[0, 1)$ — not open. Continuity plus an open *domain* does not guarantee an open *image*; compare this with the Open Mapping Theorem for holomorphic functions in complex analysis, where it *does* hold!

Problem 7.4. *Suppose $f : \mathbb{R} \to \mathbb{R}$ is defined as follows: $f(x) = x$ if x is rational, and $f(x) = x^2$ if x is irrational. Find the set of continuity points of f.*

Hint

f is continuous exactly where $x = x^2$, i.e., at $x = 0$ and $x = 1$. Everywhere else the rational and irrational "branches" pull apart, since both $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ are dense.

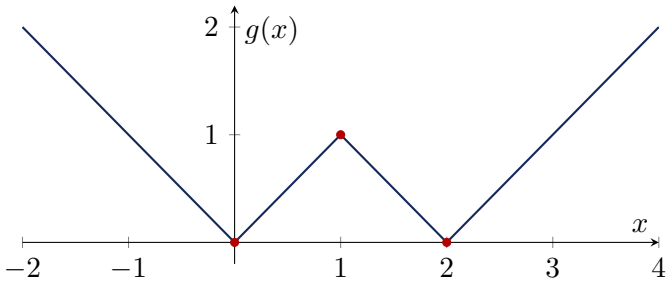
Problem 7.5. *Let $g : \mathbb{R} \to \mathbb{R}$ be defined as follows:*

$$g(x) = \begin{cases} -x, & x \le 0 \\ x, & 0 \le x \le 1 \\ 2 - x, & 1 \le x \le 2 \\ x - 2, & 2 \le x \end{cases}$$

Find the set of points of $\mathbb{R}$ where g is not differentiable.

Hint

Check the corner points $x = 0, 1, 2$ using one-sided derivatives; g is a piecewise-linear "zig-zag," so it fails to be differentiable exactly at the joints where the slope jumps.



The graph of g from Problem 19: a triangular zig-zag. The three marked corners at $x = 0, 1, 2$ are exactly where left- and right-derivatives disagree.

Problem 7.6. *For each $x \in [0,1]$ let $f(x) = x$ if x is rational, and $f(x) = 1 - x$ if x is irrational. Show that:*

(a) f is continuous only at the point $x = \frac{1}{2}$.

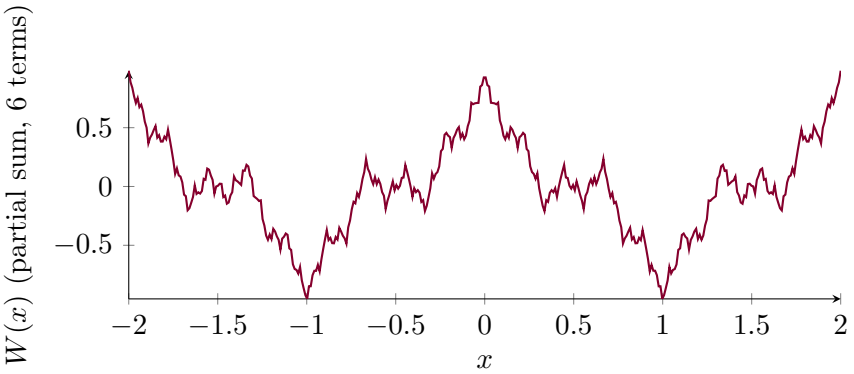
(b) f assumes every value in $[0,1]$.

:) Pun break

Why do continuous-but-not-differentiable functions make bad comedians? Their jokes always have a sharp *corner* you didn't see coming.

Companion fact — the ultimate corner case

Problems 19 and 24 deal with finitely many corners. Push this to the extreme and you get the **Weierstrass function**: a function continuous everywhere on $\mathbb{R}$ but differentiable *nowhere*. It's built as an infinite sum of increasingly wiggly cosine waves, $W(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x)$ for suitable $a \in (0,1)$, b odd. When Weierstrass first presented this in 1872, it shocked mathematicians who assumed continuous functions must be differentiable "almost everywhere."



A finite (6-term) partial sum of a Weierstrass-type series $\sum a^n \cos(b^n \pi x)$. As more terms are added the curve gets rougher at every scale — continuous, yet nowhere smooth enough to have a tangent line.

8 Difference Quotients: A Closer Look

Problem 8.1. *Let f be defined on the interval (a,b) and assume $x \in (a,b)$. Consider the two statements:*

(i) $\lim_{h\to 0} |f(x+h) - f(x)| = 0$

(ii) $\lim_{h\to 0} \big[f(x+h) - f(x-h)\big] = 0$

(a) *Show that (i) always implies (ii).*

(b) *Give an example in which (ii) holds but (i) does not hold.*

Hint

For part (b), try a function with a jump discontinuity that is symmetric about x, such as $f(y) = \operatorname{sgn}(y - x)$ near the point x — the symmetric difference can cancel out a jump that a one-sided limit would still detect.

9 Rational and Real Exponents

Refresher

For $a > 0$ and rational $r = p/q$ (lowest terms), $a^r := (a^p)^{1/q}$. For real exponents, one instead defines $a^x := e^{x \log a}$, using the exponential and logarithm built from calculus/series. Problem 20 asks you to check these two definitions of a^r agree on the rationals — a necessary sanity check before trusting a^x for irrational x.

Problem 9.1. *Let $a > 0$ be given. For a positive rational r, define $a^r = (a^p)^{1/q}$ where $r = p/q$ and p, q have no common factor. Also define $a^r = e^{r \log a}$. Show that the two definitions coincide.*

Problem 9.2. *Can you define the function x^x for $x > 0$? If you can, is the function differentiable at any $x_0 > 0$? If it is, find the derivative at x_0.*

Hint

Write $x^x = e^{x \log x}$ for $x > 0$. Differentiating via the chain rule, $\dfrac{d}{dx}(x^x) = x^x(\log x + 1)$.

:) Pun break

x^x walked into therapy and said, "I keep raising myself to my own expectations, and frankly, it's exhausting."

10 Uniform Continuity

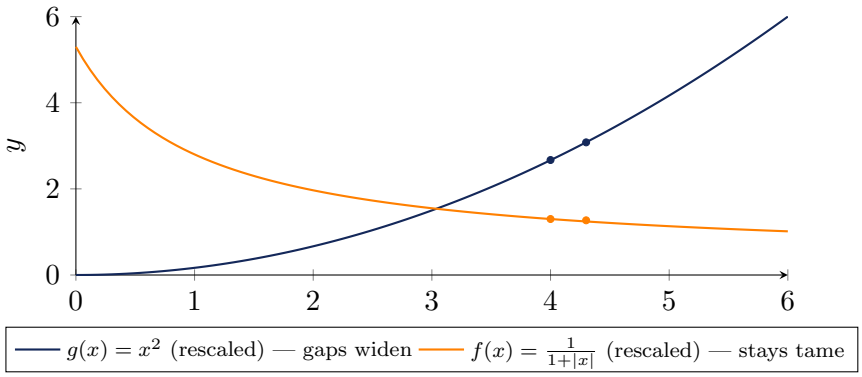
Problem 10.1. *Consider the functions $f(x) = \dfrac{1}{1+|x|}$, $g(x) = x^2$, $x \in \mathbb{R}$. Show that f is uniformly continuous on $\mathbb{R}$ but g is not.*

Hint

For f: it's bounded with a bounded derivative-like control (Lipschitz-type estimate) on all of $\mathbb{R}$. For g: take $x_n = n$, $y_n = n + \frac{1}{n}$; then $|x_n - y_n| \to 0$ but $|g(x_n) - g(y_n)| = 2 + \frac{1}{n^2} \to 2 \not\to 0$, violating uniform continuity.

Companion fact

A useful package deal: any function that is continuous on a *compact* set (like a closed bounded interval $[a, b]$) is automatically uniformly continuous there — no extra work needed. Uniform continuity only becomes a genuinely separate question on unbounded or open domains, which is exactly the setting of Problem 22.



Same tiny gap in x (schematic, curves rescaled for display): $g(x) = x^2$ stretches it into an ever-larger output gap, while $f(x) = 1/(1 + |x|)$ keeps output gaps under control everywhere — the visual essence of Problem 22.

11 Power Series for sin and cos

Problem 11.1. *Consider the following two infinite series:*

$$S(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n-1}}{(2n-1)!},$$

$$C(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}.$$

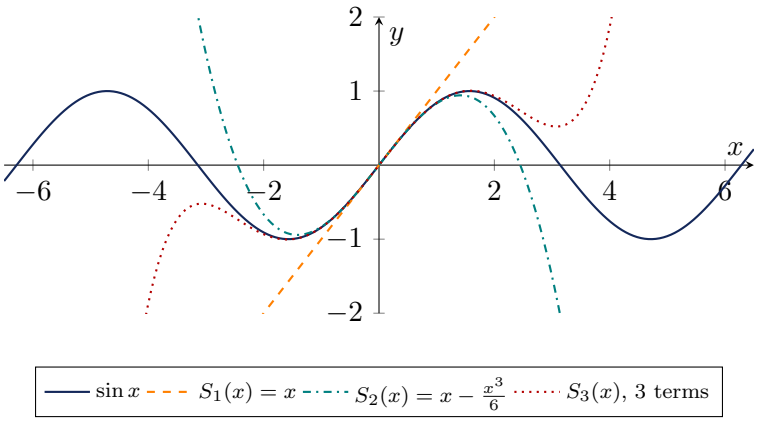
Show that both the series are absolutely convergent everywhere on $\mathbb{R}$.

Hint

Use the ratio test on $\sum |x|^{2n-1}/(2n-1)!$ (and similarly for $C(x)$): the ratio of consecutive terms tends to 0 for every fixed x, regardless of how large $|x|$ is — factorial growth always wins.

Companion fact — the payoff

Once absolute convergence is known, $S(x)$ and $C(x)$ can be *defined* as $\sin x$ and $\cos x$ (this is exactly how a rigorous analysis course builds trigonometric functions from scratch, sidestepping any appeal to "angles"!). From here, term-by-term differentiation of power series gives $S'(x) = C(x)$ and $C'(x) = -S(x)$, and the identity $S(x)^2 + C(x)^2 = 1$ can be proven purely algebraically from the series — no protractor required.



The partial sums $S(x)$ from Problem 23 converging to $\sin x$: each extra term hugs the true curve over a wider range around 0.

:) Pun break

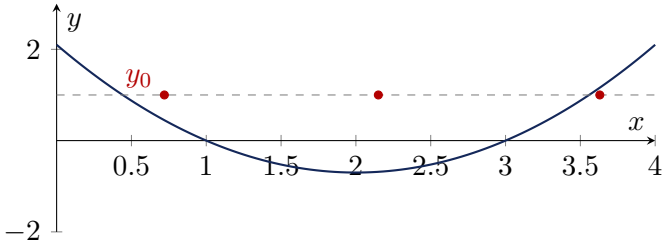
Why did $\sin x$ and $\cos x$ stay friends for life? Because no matter how their phases shifted, they always remained *a period apart* and never lost their identity: $\sin^2 x + \cos^2 x = 1$.

12 Big Theorems for Your Toolkit (Extra Material)

None of these three theorems appear explicitly on the original sheet, but they are exactly the tools that make problems like 17, 18, 19, and 24 feel natural rather than mysterious.

Intermediate Value Theorem (IVT)

If $f : [a, b] \to \mathbb{R}$ is continuous and y_0 lies between $f(a)$ and $f(b)$, then there exists $c \in [a, b]$ with $f(c) = y_0$. In particular, a continuous function cannot "jump over" a value — which is exactly why Problem 25(b) (every value in $[0, 1]$ is attained) is plausible despite f being discontinuous almost everywhere: the rational and irrational branches conspire together to still sweep out the full range.



IVT in action: a continuous curve meets a horizontal target level y_0 at least once between any two points where it's above and below y_0.

Extreme Value Theorem (EVT)

If $f : [a, b] \to \mathbb{R}$ is continuous on a closed, bounded interval, then f attains both a maximum and a minimum value on $[a, b]$. This theorem needs *compactness* of the domain — exactly the notion from Problem 4 and Section 2 — and fails the moment you switch to an open interval (e.g., $f(x) = x$ on $(0, 1)$ attains neither a max nor a min).

Mean Value Theorem (MVT)

If f is continuous on $[a, b]$ and differentiable on (a, b), there exists $c \in (a, b)$ with $f'(c) = \dfrac{f(b) - f(a)}{b - a}$. This is the natural next stop after Problem 24's difference-quotient investigation — MVT is precisely the statement that a genuine derivative (not just a symmetric difference quotient) exists somewhere in between.

:) Pun break

The Mean Value Theorem walked into a bar and ordered "whatever the average of everyone else's drink is." The bartender said, "that's oddly specific, but I can find you a point where that's exactly what you get."

12.1 Cheat Sheet: Five Definitions Worth Keeping Side by Side

Notion	**Definition (for $f : A \to \mathbb{R}$, or a sequence, as appropriate)**
Continuity at a	$\forall \varepsilon > 0\, \exists \delta > 0$ (depending on ε *and* a) s.t. $\lvert x - a\rvert < \delta \Rightarrow \lvert f(x) - f(a)\rvert < \varepsilon$.
Uniform continuity	$\forall \varepsilon > 0\, \exists \delta > 0$ (depending on ε *only*) s.t. $\lvert x - y\rvert < \delta \Rightarrow \lvert f(x) - f(y)\rvert < \varepsilon$ for *all* $x, y \in A$.
Lipschitz continuity	$\exists L > 0$ s.t. $\lvert f(x) - f(y)\rvert \le L\lvert x - y\rvert$ for all $x, y \in A$. (Strictly stronger than uniform continuity.)
Cauchy sequence	$\forall \varepsilon > 0\, \exists N$ s.t. $m, n \ge N \Rightarrow \lvert x_n - x_m\rvert < \varepsilon$.
Compactness	Every open cover of A has a finite subcover $\iff$ (in $\mathbb{R}$) A is closed and bounded.

Why the cheat sheet matters

Notice the chain of implications running through Problems 13, 14, and 22:

$$\text{Lipschitz} \;\Rightarrow\; \text{uniformly continuous} \;\Rightarrow\; \text{continuous},$$

but none of the arrows reverse in general. The classic witness for the first arrow is $f(x) = \sqrt{x}$ on $[0, \infty)$: it is uniformly continuous there, yet *not* Lipschitz, since its derivative $1/(2\sqrt{x})$ blows up as $x \to 0^+$. For the second arrow, $g(x) = x^2$ from Problem 22 is continuous everywhere but not uniformly so on all of $\mathbb{R}$, since it is only *locally* (not globally) Lipschitz — its slope $2x$ grows without bound.

13 Extra Practice: A Few Bonus Problems

Not from the original sheet — companions to the topics above, for extra practice.

Problem 13.1 (Bonus)**.** *Show directly from the ε–δ definition that $f(x) = x^2$ is uniformly continuous on $[0, 10]$ but use Problem 22's method to show it fails on all of $\mathbb{R}$.*

Problem 13.2 (Bonus)**.** *Let $\{a_n\}$ be a sequence with $\limsup |a_n|^{1/n} = L$. State (no proof needed) the Cauchy–Hadamard formula for the radius of convergence of $\sum a_n x^n$, and use it to independently confirm the convergence found in Problem 23.*

Problem 13.3 (Bonus)**.** *Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ that is discontinuous at every point of $\mathbb{R}$. (Compare this with the Dirichlet function, $\mathbf{1}_{\mathbb{Q}}$, the indicator of the rationals.)*

Problem 13.4 (Bonus)**.** *Using the Bolzano–Weierstrass theorem (see the bonus box after Problem 7), prove the Heine–Borel theorem in one direction: every closed, bounded subset of $\mathbb{R}$ is compact.*

Closing thought. Real analysis is the art of making precise everything calculus lets you get away with informally. Every ε deserves its δ, every limit deserves its proof, and — as this sheet shows — every serious theorem deserves at least one accompanying pun. Happy proving!